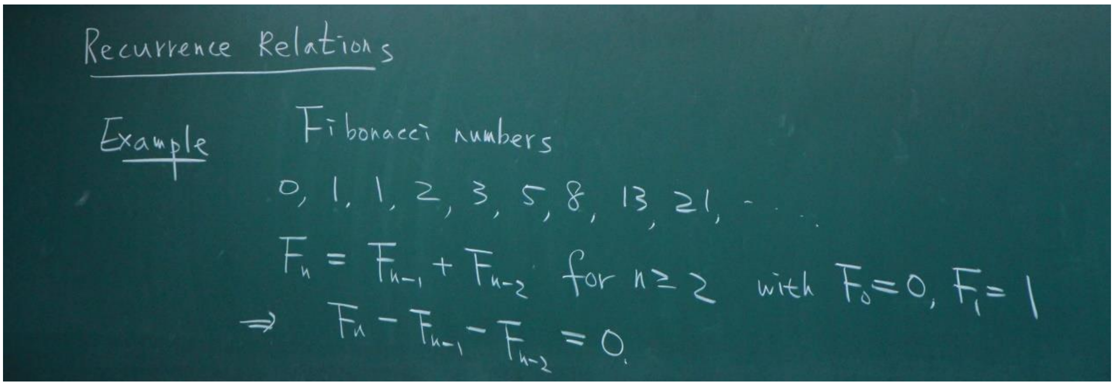
【10920 趙啟超教授離散數學 / 第 12 堂版書】



$0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 =$

$\Rightarrow \quad F_n - F_{n-1} - F_{n-2} = 0.$

Homogeneous Recurrence Relations

In general.

$$\boxed{\begin{array}{l} a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0 \\ \text{with } a_0 = A_0,\ a_1 = A_1,\ \cdots,\ a_{k-1} = A_{k-1}. \end{array}}$$

Recurrence Relations

Example Fibonacci numbers

$0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$

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k-th order linear homogeneous recurrence relation (HRR)
with constant coefficients

or k-th order linear homogeneous difference equation (HDE)
with constant coefficients

with constant coefficients

<u>Example</u> Solve $a_{n+1} - 5a_n + 6a_{n-1} = 0$ for $n \geq 2$
with $a_1 = 1$, $a_2 = 5$.

Try $a_n = r^n$.

$$\Rightarrow \quad r^{n+1} - 5r^n + 6r^{n-1} = 0$$

$$\Rightarrow \quad r^{n-1}(r^2 - 5r + 6) = 0$$

$$\Rightarrow \quad \underline{r^2 - 5r + 6 = 0} \longrightarrow \text{characteristic equation}$$

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$$\Rightarrow \quad (r-2)(r-3) = 0$$

$$\Rightarrow \quad r = 2, 3$$

Hence both $a_n = 2^n$ and $a_n = 3^n$ are solutions.

Since the equation is linear

$$a_n = \alpha_1 2^n + \alpha_2 3^n \text{ is also a solution}$$

for any constants α_1 and α_2.

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For initial conditions,

$$1 = a_1 = 2\alpha_1 + 3\alpha_2$$

$$5 = a_2 = 4\alpha_1 + 9\alpha_2$$

$$\Rightarrow \alpha_1 = -1, \quad \alpha_2 = 1$$

$\therefore$ $a_n = -2^n + 3^n$, for $n \geq 1$. For example, $a_6 = -2^6 + 3^6 = 665$.

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Example Solve $F_n - F_{n-1} - F_{n-2} = 0$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$.

characteristic eq. $r^2 - r - 1 = 0$

$$\Rightarrow \quad r = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ General solution $= \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

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$$\Rightarrow \quad \alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}.$$

$\therefore$ $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. for $n \geq 0$

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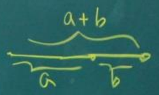
$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

$1.618 \approx \frac{1+\sqrt{5}}{2}$ golden ratio



$\frac{a+b}{a} = \frac{a}{b}$

Example $a_n - 4a_{n-1} + 4a_{n-2} = 0$

characteristic eq. $r^2 - 4r + 4 = 0$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r = 2, 2$$

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However, $a_n = n2^n$ is also a solution.

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$\overbrace{\underbrace{\qquad}_{a}\underbrace{\qquad}_{b}}^{a+b}$ $\qquad \frac{a+b}{a} = \frac{a}{b}$

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since

$$\begin{aligned}
& a_n - 4a_{n-1} + 4a_{n-2} \\
&= n2^n - 4(n-1)2^{n-1} + 4(n-2)2^{n-2} \\
&= 2^{n-2}\left[n\cdot 2^2 - 4(n-1)\cdot 2 + 4(n-2) \right] \\
&= 2^{n-2}\left[n(4-8+4) + (8-8) \right] \\
&= 0.
\end{aligned}$$

$\therefore$ The general solution is $\alpha_1 2^n + \alpha_2 n 2^n$.

$$= 0.$$

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In general, if the characteristic equation is $(r-r_0)^m$, then

$$a_n = r_0^n,\quad n r_0^n,\quad n^2 r_0^n,\quad \cdots,\quad n^{m-1} r_0^n$$

are all solutions to the HRR.

since $a_n - 4a_{n-1} + 4a_{n-2}$

$$= n 2^n - 4(n-1) 2^{n-1} + 4(n-2) 2^{n-2}$$

$$= 2^{n-2}\left[n\cdot 2^2 - 4(n-1)\cdot 2 + 4(n-2) \right]$$

$$= 2^{n-2}\left[n(4-8+4) + (8-8) \right]$$

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Characteristic Equations with Complex Roots

Example $a_n - 2a_{n-1} + 2a_{n-2} = 0$ for $n \ge 2$

with $a_0 = 1,\ a_1 = 2.$

characteristic equation

$$r^2 - 2r + 2 = 0$$

$$\Rightarrow \quad r = 1 \pm j \quad (\text{where } j = \sqrt{-1})$$

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$$\Rightarrow \quad r = \sqrt{2}\, e^{j\frac{\pi}{4}} \text{ or } \sqrt{2}\, e^{-j\frac{\pi}{4}}. \qquad (e^{j\theta} = \cos\theta + j\sin\theta)$$

The general solution is

$$a_n = \alpha_1 \left(\sqrt{2}\, e^{j\frac{\pi}{4}}\right)^n + \alpha_2 \left(\sqrt{2}\, e^{-j\frac{\pi}{4}}\right)^n$$

$$= \alpha_1\, 2^{\frac{n}{2}}\, e^{j\frac{n\pi}{4}} + \alpha_2\, 2^{\frac{n}{2}}\, e^{-j\frac{n\pi}{4}}$$

$$= 2^{\frac{n}{2}} \left[(\alpha_1 + \alpha_2) \cos\frac{n\pi}{4} + j(\alpha_1 - \alpha_2) \sin\frac{n\pi}{4} \right]$$

Characteristic Equations with Complex Roots

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$$= \alpha_1 2^{\frac{n}{2}} e^{j\frac{n\pi}{4}} + \alpha_2 2^{\frac{n}{2}} e^{-j\frac{n\pi}{4}}$$

$$= 2^{\frac{n}{2}} \left[\underbrace{(\alpha_1 + \alpha_2)}_{=\beta_1} \cos\frac{n\pi}{4} + \underbrace{j(\alpha_1 - \alpha_2)}_{=\beta_2} \sin\frac{n\pi}{4} \right]$$

for (possibly complex) constants α_1 and α_2.

A new form of the general solution

$$a_n = \beta_1 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + \beta_2 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

for real constants β_1 and β_2.

For initial conditions

$$1 = a_0 = \beta_1$$

$$2 = a_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}}\beta_1 + \frac{1}{\sqrt{2}}\beta_2\right) = \beta_1 + \beta_2.$$

$$\Rightarrow \beta_1 = 1, \quad \beta_2 = 1.$$

$$\therefore a_n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right) \quad \text{for } n \geq 0.$$

for (possibly complex) constants α_1 and α_2.

A new form of the general solution

$$a_n = \beta_1\, 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + \beta_2\, 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

for real constants β_1 and β_2.

In general, if the roots of the characteristic equation are $\lambda e^{i\theta}$ and $\lambda e^{-i\theta}$, then the general solution to the HRR can be written as

$$a_n = \beta_1 \lambda^n \cos n\theta + \beta_2 \lambda^n \sin n\theta$$

$\lambda e^{j\theta}$ and $\lambda e^{-j\theta}$, then the general solution to the HRR can be written as

$$a_n = \beta_1 \lambda^n \cos n\theta + \beta_2 \lambda^n \sin n\theta$$

Nonhomogeneous Recurrence Relations

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = f_n$$

forcing sequence

kth-order linear nonhomogeneous (inhomogeneous) recurrence relation (NRR) with constant coefficients

or kth-order linear nonhomogeneous difference equation (NDE) with constant coefficients.

Let a_n be any solution to this NRR and p_n be a particular solution

Then

$$a_n - c_1 a_{n-1} - \cdots - c_k a_{n-k} = f_n$$

$$p_n - c_1 p_{n-1} - \cdots - c_k p_{n-k} = f_n$$

$$\Rightarrow (a_n - p_n) - c_1 (a_{n-1} - p_{n-1}) - \cdots - c_k (a_{n-k} - p_{n-k}) = f_n - f_n = 0$$

and p_n be a particular solution

Then $a_n - c_1 a_{n-1} - \cdots - c_k a_{n-k} = f_n$

$p_n - c_1 p_{n-1} - \cdots - c_k p_{n-k} = f_n$

$\Rightarrow (a_n - p_n) - c_1(a_{n-1} - p_{n-1}) - \cdots - c_k(a_{n-k} - p_{n-k}) = f_n - f_n = 0$

We have $z_n = a_n - p_n$ satisfies the associated HRR

$z_n - c_1 z_{n-1} - \cdots - c_k z_{n-k} = 0.$

Conversely, let z_n be any solution the HRR and p_n be a particular solution to the NRR.

Then $z_n - c_1 z_{n-1} - \cdots - c_k z_{n-k} = 0$

$p_n - c_1 p_{n-1} - \cdots - c_k p_{n-k} = f_n$

$\Rightarrow (z_n + p_n) - c_1(z_{n-1} + p_{n-1}) - \cdots - c_k(z_{n-k} + p_{n-k}) = f_n$

We have $a_n = z_n + p_n$ is a solution to the NRR.

$\therefore$ The general solution to a given NRR is obtained by adding the general solution to the associated HRR and any particular solution to the NRR.

<u>Example</u> $a_n - a_{n-1} - 2a_{n-2} = 1, \quad n \geq 3$

with $a_1 = 1, \; a_2 = 3.$

HRR: $a_n - a_{n-1} - 2a_{n-2} = 0$

characteristic eq. $r^2 - r - 2 = 0$

$\Rightarrow r = 2, -1$

The general solution to the associated HRR is

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Guess a particular to the NRR:

$$p_n = B.$$

$$\Rightarrow B - B - 2B = 1 \Rightarrow B = -\frac{1}{2}.$$

$\therefore$ The general solution to the NRR is

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n - \frac{1}{2}.$$

For initial conditions,

$$1 = a_1 = 2\alpha_1 - \alpha_2 - \frac{1}{2}$$
$$3 = a_2 = 4\alpha_1 + \alpha_2 - \frac{1}{2}$$
$$\Rightarrow \alpha_1 = \frac{5}{6}, \quad \alpha_2 = \frac{1}{6}$$
$$\therefore \quad a_n = \frac{5}{6}\cdot 2^n + \frac{1}{6}(-1)^n - \frac{1}{2}, \quad \text{for } n \geq 1$$

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